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On the fluctuations of the Casimir force

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Abstract. Standard statistical and quantum physics are used to analyse the fluctuations of the Casimir stress S (normal force per unit area) exerted on a flat perfect mirror (conductor) by the zero-point electromagnetic fields in adjacent space, partly in order to help dispel the apologetics that often befog zero-point effects generally. S is measurable only when averaged ($\bar{S}$) over finite times T, and also ($\bar{\bar{S}}$) over finite areas of typical linear dimensions a. With only one mirror, the mean-square deviations are $\langle \Delta\bar{S}^2\rangle = \text{constant} \times \hbar^2/c^6T^8$, the value of such constants depending on how the apparatus averages over time; if $a \ll cT$, then $\langle \Delta\bar{\bar{S}}^2\rangle = \langle \Delta\bar{S}^2\rangle$. If, unrealistically, $a \gg cT$, then $\langle \Delta\bar{\bar{S}}^2\rangle = \text{constant} \times \hbar^2/c^4a^2T^6$. On one of a pair of parallel mirrors a distance L apart, if $cT \gg a$, $cT \gg L$, then $\langle \Delta\bar{\bar{S}}^2\rangle = \langle \Delta\bar{S}^2\rangle = \text{constant} \times \hbar^2/c^4L^2T^6$. Mirror transparency at frequencies well above $1/T$ has negligible effect. An appendix outlines the mathematical problems, mostly unsolved, met in attempts to evaluate the full probability distributions underlying such mean-square deviations.

1. Introduction and conclusions

The Casimir force is the attraction $\pi^2\hbar c/240L^4$ per unit area† between two flat, parallel and indefinitely extended perfect mirrors (i.e. perfect conductors) a distance L apart. Numerically it reads 1.30×10^{-18} $(\mathrm{cm/L})^4$ dyne cm^{-2}. It is the prime manifestation of the zero-point motions of quantized fields, and like the weather (Warner 1897) in that everyone talks but nobody does anything about it: see, for example, Plunien *et al* (1986) and Fulling (1989) for recent discussions, and Sparnaay (1958) for the only measurement on metals (rather than insulators). Even the questions raised in such discussions have remained quite remarkably restricted. Here we extend their scope by considering the fluctuations of the Casimir force when averaged not over the entire surface of a nominally infinite mirror, but only over a part having typical linear dimensions a, and area A of order a^2; and also, as in practical measurements it must be, over some finite time interval T. One of our aims is to de-mystify zero-point fields *in general*, by subjecting their manifestations to the kinds of analyses that are routine in quantum mechanics and in the kinetic theory of gases. The gas we have in mind is the photon gas, albeit at zero temperature, and although analogies with non-relativistic gases prove more fruitful in suggesting questions rather than answers.

Envisage then a piston forming part but movable independently of the rest of a finitely thick but infinite-area perfect mirror normal to the z axis, and consider the

† Henceforth we generally use natural units $\hbar = 1 = c$, and unrationalized Gaussian units for the Maxwell field. It proves crucial that $\boldsymbol{E}$ and $\boldsymbol{B}$ then have dimensions $1/(\text{time})^2 = 1/(\text{length})^2$.

impulse delivered to the piston over an interval by the zero-point fields†. The force they exert per unit area is given by the 33 component of the stress tensor‡ $S_{ij} = [E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(\boldsymbol{E}^2 + \boldsymbol{B}^2)]/4\pi$; on the mirror, since the boundary conditions enforce $\boldsymbol{E}_\| = 0 = B_3$, we have

$$S_{33} = \frac{(E_3^2 - \boldsymbol{B}_\|^2)}{8\pi}. \tag{1.1}$$

(The subscript $\|$ identifies vector components parallel to the surface: e.g. $\boldsymbol{r}_\| \equiv (x, y)$.) Positive S_{33} corresponds to a negative pressure, i.e. to a force directed into the vacuum.

By hindsight we postpone the surface averaging, and start, still for given (x, y) on say the right-facing mirror surface, with the time average

$$\bar{S} \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t) S_{33}(t) \tag{1.2a}$$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, f(t) = 1. \tag{1.2b}$$

(The subscripts 33 are dropped from $\bar{S}$ for brevity.) The averaging function f depends on the measuring apparatus. Its precise shape, though it does effect the final numerical coefficients, need not be chosen until later; meanwhile, on physical grounds we take f to be real and non-negative, and more specifically as a single peak of width $2T$ comparable to the experimental resolving time.

Later we shall introduce the joint time and surface average

$$\tilde{\bar{S}} \equiv \int \mathrm{d}^2 r_\|\, \phi(\boldsymbol{r}_\|) \bar{S} \tag{1.3a}$$

$$\int \mathrm{d}^2 r_\|\, \phi(\boldsymbol{r}_\|) = 1 \tag{1.3b}$$

with ϕ likewise real and non-negative, and taken as a single peak extending over a region of linear dimensions a comparable to those of the piston. However, in most practical cases the shape of ϕ, unlike the shape of f, proves irrelevant.

Note that these time and surface averages identified by overbar and tilde respectively are still operators in Hilbert space (i.e. in the photon Fock space); indeed they are precisely the random variables under study, whose quantal expectation values, vacuum (ground-state) expectation values in our case, will be denoted by angular brackets, $\langle \ldots \rangle$. Thus, the mean Casimir force is related to $\langle S \rangle = \langle \bar{S} \rangle = \langle \tilde{\bar{S}} \rangle$, and its measurable fluctuations, via $\langle \tilde{\bar{S}}^2 \rangle$, to the mean-square deviation

$$\langle \Delta \tilde{\bar{S}}^2 \rangle \equiv \langle \tilde{\bar{S}}^2 \rangle - \langle \tilde{\bar{S}} \rangle^2. \tag{1.4}$$

The layout of the rest of this paper, and its main conclusions, are as follows. Section 2 exorcises some common preconceptions which experience suggests could otherwise

† Here we consider only pistons with surface area appreciably less than that of the mirror. It remains to be investigated whether any special simplicities emerge when the piston comprises the entire mirror, i.e. an entire end wall of the system, which in realizable systems must be finite. Meanwhile, the idealization to infinitely extended mirrors greatly eases the calculations.

‡ The first to discuss the Casimir effect in terms of the stress tensor were Brown and Maclay (1969). Most others discuss it in terms of zero-point energies; this approach however does not readily lend itself to elucidating fluctuations.

much retard one's understanding of the physics. Section 3 introduces the quantized fields in the simplest Casimir-related situation, with only one mirror; it also outlines the two alternative and physically quite different roles that can be assigned to a frequency cut-off in such calculations.

Section 4 considers the force per unit area on a test piston set in a single mirror, whose mean value naturally vanishes by symmetry. In the realistic case where $a \ll cT$, it turns out that surface averaging makes no further difference to the finite-time-averaged fluctuations, and one finds

$$\langle \Delta \bar{\bar{S}}^2 \rangle = \langle \Delta \bar{S}^2 \rangle = \frac{\text{constant}}{T^8} \qquad (a \ll cT)$$

the main results being given by (4.14) and (4.10*a*, *b*). In the opposite and rather academic extreme where $a \gg cT$, one finds

$$\langle \Delta \bar{\bar{S}}^2 \rangle = \frac{\text{constant}}{a^2 T^6}$$

as given by (4.17).

Section 5 considers a similar test piston set into one of a pair of parallel mirrors, where the force per unit area has the non-zero mean value quoted at the start of this introduction, and rederived in (5.1) and (5.2). When, realistically, $cT \gg L$ and $cT \gg a$, one finds

$$\langle \Delta \bar{\bar{S}}^2 \rangle = \langle \Delta \bar{S}^2 \rangle = \frac{\text{constant}}{L^2 T^6}$$

as given by (5.4). The relative RMS fluctuations are then of order $(L/cT)^3 \ll 1$.

Finally, the appendix describes some problems one meets if, going beyond the mean-square deviations, one tries to determine the probability distributions in full. Mathematically this is manageable for the Cartesian components of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ themselves, whose distributions are Gaussians; but the results make physical sense only for finite-time-averaged fields, whose distributions have widths proportional to T^{-2}. The literature does supply formal expressions for the characteristic functions of the stress tensor or of the energy density, but these are likely to be quite awkward to evaluate, and for the finite-time-averaged quantities very difficult indeed.

2. Exorcism

It may obviate some confusing reflexes if we recall here two points from the ordinary Maxwell-Boltzmann kinetic theory of ideal gases.

First, there is at best only a very tenuous connection between, on the one hand, the 'pressure fluctuations' prescribed for a test volume V by the textbook formula $\Delta p^2 = (2k\Theta^2/3V)(\partial p/\partial\Theta)_V$ (e.g. see Reif 1965), and, on the other hand, the mean-square deviation of the finite-time-averaged force, call it $\bar{F}$, experienced by a piston of finite area A (Fowler 1936). The former, like the mean pressure itself, are bulk equilibrium properties governed by the equation of state; they are loosely analogous to the unaveraged fluctuations discussed in the appendix, and have similarly and surprisingly little relevance to measured forces. By contrast, the latter depend not merely on the (Poisson) time distribution of the molecular impacts, but also on the interaction mechanism between molecule and piston surface. For instance, if a fraction $(1-\alpha)$ of incident molecules are reflected specularly, while the rest stick and are eventually re-emitted in a statistically independent process, then $\Delta\bar{F}^2(\alpha)=(1-\alpha/2)\Delta\bar{F}^2(0)$. It is of course $\bar{F}/A$ that is loosely analogous to the measurable quantity $\bar{S}$ studied in sections 4 and 5. Thus it seems unlikely that the perfect-mirror fluctuation formulae derived there can be extended rigorously to imperfect materials in terms simply of their bulk electromagnetic response functions†, in the way that the mean stresses can be (Dzyaloshinskii *et al* 1961, Lifshitz and Pitaevskii 1980).

Second, we recall that for the ideal Maxwell–Boltzmann gas the mean-square deviation $\Delta(\bar{F}/A)^2$ is proportional to $1/AT$, simply by the law of large numbers, i.e. simply because the molecular impacts are statistically independent. By contrast, for a quantum gas, even for an ideal one, and for the photon gas in particular, the impacts are highly correlated; consequently we shall obtain the quite different dependences on a and T already anticipated in the introduction. In other words, though a simple photon-gas model of the Casimir effect, like that suggested by Milonni *et al* (1988), is suited to interpreting the mean Casimir force, it affords no immediate insight into its fluctuations.

3. Quantization in a half-space, and the role of cut-offs

For later reference we start with the quantized field operators in unbounded space:

$$E=\frac{\mathrm{i}}{2\pi}\sum_{s=1,2}\int_{-\infty}^{\infty}\mathrm{d}^3K\,\sqrt{\omega}\,a_s(\boldsymbol{K})\boldsymbol{\varepsilon}_s(\boldsymbol{K})\,\mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r}-\mathrm{i}\omega t}+\mathrm{HC} \tag{3.1}$$

and similarly for $\boldsymbol{B}$ with $\boldsymbol{\varepsilon}\to\hat{\boldsymbol{K}}\wedge\boldsymbol{\varepsilon}$; here, HC stands for Hermitean conjugate, $\omega=|\boldsymbol{K}|$, $\boldsymbol{\varepsilon}_s\cdot\boldsymbol{K}=0$, $\boldsymbol{\varepsilon}_s(\boldsymbol{K})\cdot\boldsymbol{\varepsilon}_{s'}(\boldsymbol{K})=\delta_{ss'}$, and the creation and annihilation operators a^+, a obey the usual commutation rules. Hats denote unit vectors.

For a single mirror with its right-hand surface at $z=0$, the fields in the half-space $z\geqslant 0$ may be written (e.g. see Barton and Fawcett 1988)

$$\boldsymbol{E}=\frac{\mathrm{i}}{\pi}\int_0^{\infty}\mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}^2k\,\sqrt{\omega}\bigg\{a_1(\boldsymbol{k},l)\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{z}}\sin(lz) + a_2(\boldsymbol{k},l)\bigg[\hat{\boldsymbol{k}}\frac{\mathrm{i}l}{\omega}\sin(lz)-\hat{\boldsymbol{z}}\frac{k}{\omega}\cos(lz)\bigg]\bigg\}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_{\parallel}-\mathrm{i}\omega t}+\mathrm{HC} \tag{3.2a}$$

$$\boldsymbol{B}=\frac{1}{\pi}\int_0^{\infty}\mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}^2k\frac{1}{\sqrt{\omega}}\bigg\{a_1[-\mathrm{i}\hat{\boldsymbol{z}}k\sin(lz)+\hat{\boldsymbol{k}}l\cos(lz)] + a_2(-\mathrm{i}\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{z}})\omega\cos(lz)\bigg\}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_{\parallel}-\mathrm{i}\omega t}+\mathrm{HC}. \tag{3.2b}$$

Here $\omega\equiv(k^2+l^2)^{1/2}$, and

$$[a_s(\boldsymbol{k},l),a^+_{s'}(\boldsymbol{k}',l')]=\delta_{ss'}\delta(l-l')\delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}') \tag{3.2c}$$

while a or a^+ pairs commute. Integrations over l and $\boldsymbol{k}=(k_1,k_2)$ will always be understood to run between the limits shown in (3.2). Note the option of changing variables from $(l,\boldsymbol{k})$ to the spherical-polar components (K,θ,ϕ) of $\boldsymbol{K}$, with $0\leqslant\theta\leqslant\pi/2$, so that $l=K\cos\theta$, $k=K\sin\theta$ and $\int\mathrm{d}l\,\mathrm{d}^2k\ldots=\int_0^{\infty}\mathrm{d}K\,K^2\int_0^{2\pi}\mathrm{d}\phi\int_0^1\mathrm{d}\cos\theta\ldots$. We

† For example, it is a challenge to extend the theory of the Casimir effect to black surfaces, defined as absorbing all the radiation that falls on them. As regards particles, a surface black in this sense is readily characterized by an accommodation coefficient $\alpha=1$. But no convincing and practically useful way is known of representing blackness by boundary conditions imposable on the solutions of partial differential wave equations: this has been discussed in classical diffraction theory (Kottler 1965, 1967).

shall often subsume all the normal-mode labels into λ, write the normal-mode amplitudes as $\boldsymbol{E}_\lambda(\boldsymbol{r}, t)$ and $\boldsymbol{B}_\lambda(\boldsymbol{r}, t)$, and set $\boldsymbol{E} = \Sigma_\lambda a_\lambda \boldsymbol{E}_\lambda + \text{HC}$, $\boldsymbol{B} = \Sigma_\lambda a_\lambda \boldsymbol{B}_\lambda + \text{HC}$. Thus, with a single mirror, $\Sigma_\lambda \ldots$ stands for $\Sigma_{s=1,2} \int \mathrm{d}l\, \mathrm{d}^2k \ldots$. The vacuum is written $|0\rangle$, with $\langle 0| \ldots |0\rangle \equiv \langle \ldots \rangle$, and one- and two-photon states as $|\lambda\rangle$ and $|\lambda\lambda'\rangle$.

On the mirror surface the surviving fields featured in S_{33} reduce to

$$E_3(z=0) = -\frac{\mathrm{i}}{\pi} \int \mathrm{d}l\, \mathrm{d}^2k \frac{1}{\sqrt{\omega}} a_2 k\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\parallel - \mathrm{i}\omega t} + \text{HC} \tag{3.3a}$$

$$\boldsymbol{B}_\parallel(z=0) = \frac{1}{\pi} \int \mathrm{d}l\, \mathrm{d}^2k \frac{1}{\sqrt{\omega}} [a_1 \hat{\boldsymbol{k}} l - a_2(\mathrm{i}\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{z}})\omega]\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\parallel - \mathrm{i}\omega t} + \text{HC}. \tag{3.3b}$$

Under Σ_λ, with summands quadratic in the normal-mode amplitudes, we introduce, whenever it is mathematically convenient, a frequency cut-off chosen somewhat arbitrarily as $\exp(-\omega/\Omega)$. We shall just as freely drop such factors when they are no longer needed. The cut-off can be called upon to play two physically quite different roles, and it is essential to distinguish between them very clearly (e.g. see Barton and Fawcett 1988, appendix F).

First, in unbounded space, and also regarding mirror-related effects as long as the idealization to perfect mirrors remains unquestioned, the cut-off serves as a purely auxiliary mathematical device, and physical significance attaches only to quantities well behaved in the 'no-cut-off limit' $\Omega \to \infty$. (The import of these remarks may become clearer in the appendix.)

Alternatively, in mirror-related effects calculated via (3.3), one can try to interpret such sums with finite cut-off as very rough and provisional order-of-magnitude estimates for imperfect mirrors. To this end one might equate Ω to the plasma frequency ω_p of metallic mirrors, because at frequencies well above ω_p such materials become effectively transparent; this in turn renders the boundary conditions inoperative, whence in one way or another contributions from opposite sides of the mirror tend to cancel. The present paper is almost wholly confined to perfect mirrors, with only a few asides on this second interpretation of the cut-off: we hope to discuss more realistic models for mirrors more sensibly elsewhere. Meanwhile, in order-of-magnitude estimates we shall set $\Omega \approx \omega_\mathrm{p} \approx 1.65 \times 10^{16}\ \mathrm{s}^{-1}$, the plasma frequency of copper.

4. Fluctuations with a single mirror

It is easy to see that on the mirror equations (3.3) with a cut-off entail

$$\langle E_3^2 \rangle = \sum_\lambda |\langle \lambda | E_3 | 0 \rangle|^2 = \frac{1}{\pi^2} \int \mathrm{d}l \int \mathrm{d}^2k \frac{1}{\omega} k^2\, \mathrm{e}^{-\omega/\Omega} \tag{4.1a}$$

$$\langle \boldsymbol{B}_\parallel^2 \rangle = \sum_\lambda |\langle \lambda | \boldsymbol{B}_\parallel | 0 \rangle|^2 = \frac{1}{\pi^2} \int \mathrm{d}l \int \mathrm{d}^2k \frac{1}{\omega} (l^2 + \omega^2)\, \mathrm{e}^{-\omega/\Omega} \tag{4.1b}$$

whence

$$\langle S \rangle \equiv \langle S_{33} \rangle = \frac{1}{8\pi^3} \int \mathrm{d}l \int \mathrm{d}^2k \frac{1}{\omega} [k^2 - (l^2 + \omega^2)]\, \mathrm{e}^{-\omega/\Omega} \tag{4.2a}$$

$$= \frac{1}{8\pi^3} \int_0^\infty \mathrm{d}K\, K^3\, \mathrm{e}^{-K/\Omega} 2\pi \int_0^1 \mathrm{d}\cos\theta (\sin^2\theta - \cos^2\theta - 1)$$

$$= -\frac{1}{6\pi^2} \int_0^\infty \mathrm{d}K\, K^3\, \mathrm{e}^{-K/\Omega} = -\frac{\Omega^4}{\pi^2}. \tag{4.2b}$$

The minus sign specifies a positive pressure. That $\langle E^2\rangle$, $\langle B^2\rangle$ and $\langle S\rangle$ must be proportional to Ω^4 follows on dimensional grounds alone. The provisional estimate $\Omega \approx \omega_p$ for imperfect mirrors (see section 3) yields $\langle S\rangle \approx 2.9\times 10^5$ dyne cm^{-2} = 29 atm, far in excess not only of the net Casimir force per unit area between parallel mirrors at achievable separations, but also of the ordinary radiation pressure at finite laboratory temperatures, which is only $\pi^2(k\Theta)^4/45 = 2\times 10^{-5}(\Theta/300)^4$ dyne cm^{-2}.

Since $\langle S\rangle$ is independent of t and $\boldsymbol{r}$, averaging makes no difference, as already anticipated just above (1.4). But its contribution to the vacuum expectation-value of the force on our test piston is cancelled identically by an equal and opposite contribution from the opposite face of the piston, so that, for a single mirror, the total mean force vanishes. By symmetry, the same is true for imperfect mirrors.

By contrast, no such cancellation occurs in the mean-square values of S or of its various averages. On the contrary, the right-hand and the left-hand half-spaces separated by our perfect mirror constitute completely disjoint electromagnetic systems; the fields in these half-spaces fluctuate independently of each other; and the mean-square values of the stresses and forces on the two sides simply add. *Right from the outset therefore when treating fluctuations we are deprived of the mechanism which eliminates divergences from the traditional calculations of the mean Casimir force itself*, as at the start of section 5 below. (For imperfect mirrors, fluctuations in the two half-spaces will be effectively correlated above but not below the plasma frequency.)

Nevertheless it proves convenient to begin by writing down the mean-square deviation of the unaveraged stress S_{33} itself, even though in the present context it is a purely auxiliary quantity. One finds straightforwardly, first the requisite two-photon matrix elements from (1.1) and (3.3), namely

$$\langle l\boldsymbol{k}1, l'\boldsymbol{k}'1|S_{33}|0\rangle = -\frac{1}{4\pi^3}\frac{ll'}{\sqrt{\omega\omega'}}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}')\exp[\ldots]$$

$$\langle l\boldsymbol{k}1, l'\boldsymbol{k}'2|S_{33}|0\rangle = -\frac{\mathrm{i}}{4\pi^3}\frac{l\omega'}{\sqrt{\omega\omega'}}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'\wedge\hat{\boldsymbol{z}})\exp[\ldots]$$

$$\langle l\boldsymbol{k}2, l'\boldsymbol{k}'2|S_{33}|0\rangle = \frac{1}{4\pi^3}\frac{1}{\sqrt{\omega\omega'}}[-kk'+\omega\omega'(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{z}})\cdot(\hat{\boldsymbol{k}}'\wedge\hat{\boldsymbol{z}})]\exp[\ldots]$$

where the exponent is $[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}_\parallel + \mathrm{i}(\omega+\omega')t]$; and thence

$$\langle\Delta S^z\rangle \equiv \langle S_{33}^2\rangle - \langle S_{33}\rangle^2 \tag{4.3a}$$

$$= \sum_{\lambda\lambda'}|\langle\lambda\lambda'|S_{33}|0\rangle|^2 \tag{4.3b}$$

$$= \frac{1}{16\pi^6}\int \mathrm{d}l\,\mathrm{d}^2k\int \mathrm{d}l'\,\mathrm{d}^2k'\frac{1}{\omega\omega'}\mathrm{e}^{-(\omega+\omega')/\Omega}$$
$$\times\{l^2l'^2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}')^2 + l^2\omega'^2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'\wedge\hat{\boldsymbol{z}})^2 + [-kk'+\omega\omega'(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{z}})\cdot(\hat{\boldsymbol{k}}'\wedge\hat{\boldsymbol{z}})]^2\} \tag{4.3c}$$

$$= \frac{7}{24\pi^4}\int_0^\infty \mathrm{d}K\,K^3\,\mathrm{e}^{-K/\Omega}\int_0^\infty \mathrm{d}K'\,K'^3\,\mathrm{e}^{-K'/\Omega} \tag{4.3d}$$

$$= \frac{21}{2\pi^4}\Omega^8. \tag{4.3e}$$

Equation (4.3b) stems from the selection rules for photon numbers; in (4.3c) we have inserted cut-offs as explained in section 3; and in (4.3d) we have changed variables, and performed the integrations over the polar angles of $\boldsymbol{K}$ and $\boldsymbol{K}'$.

For what it is worth (granted finite Ω), (4.3e) and (4.2b) yield $\langle \Delta S^2\rangle^{1/2}/|\langle S\rangle| = 3$, so that the unaveraged stress fluctuates very strongly.

Consider next the time-average $\bar{S}$ defined in (1.2), and introduce the Fourier transform $g(\nu)$ of $f(t)$:

$$g(\nu) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \mathrm{e}^{\mathrm{i}\nu t}. \tag{4.4}$$

It follows immediately that

$$\langle \Delta \bar{S}^2\rangle \equiv \langle \bar{S}^2\rangle - \langle \bar{S}\rangle^2 \tag{4.5a}$$

$$= \sum_{\lambda\lambda'} |\langle \lambda\lambda'|S_{33}|0\rangle|^2 |g(\omega+\omega')|^2. \tag{4.5b}$$

Hence, by comparison with (4.3d), we find

$$\langle \Delta \bar{S}^2\rangle = \frac{7}{24\pi^4} \int_0^\infty \mathrm{d}K\, K^3\, \mathrm{e}^{-K/\Omega} \int_0^\infty \mathrm{d}K'\, K'^3\, \mathrm{e}^{-K/\Omega} |g(K+K')|^2 \tag{4.5c}$$

$$= \frac{1}{480\pi^4} \int_0^\infty \mathrm{d}\xi\, \xi^7 |g(\xi)|^2\, \mathrm{e}^{-\xi/\Omega}. \tag{4.5d}$$

(4.5d) follows after changing variables from K, K' to $\xi \equiv (K+K')$ and $\eta \equiv (K-K')$. The factor $|g|^2$ evidently mimicks a cut-off, and we shall see presently that with realistic f, g the cut-off can now be dropped†; $\langle \Delta\bar{S}^2\rangle$ is then proportional to the seventh moment of the spectral intensity $|g|^2$ of the time-averaging function f.

Mathematically simple candidates for f include the square step

$$f(t) = \frac{1}{2T}\, \theta(T - |t|) \tag{4.6a}$$

$$g(\nu) = \frac{\sin(\nu T)}{\nu T} \tag{4.6b}$$

the Gaussian

$$f(t) = \frac{1}{\sqrt{\pi}\, T} \exp\left(-\frac{t^2}{T^2}\right) \tag{4.7a}$$

$$g(\nu) = \exp\left(-\nu^2 \frac{T^2}{4}\right) \tag{4.7b}$$

the Lorentzian

$$f(t) = \frac{(T/\pi)}{(t^2+T^2)} \tag{4.8a}$$

$$g(\nu) = \exp(-|\nu| T) \tag{4.8b}$$

† It makes no difference if the cut-off is retained in such cases because $|g(\xi)|^2$, if it falls fast enough to ensure convergence, falls much faster than does $\exp(-\xi/\Omega)$. For example, if we do keep the factor $\exp(-\xi/\Omega)$ in (4.5d) alongside the Lorentzian $|g|^2 = \exp(-2\xi T)$, the only effect on (4.10) is to replace T there by $(T + 1/2\Omega)$. But even the least imaginable resolving time T for measuring forces will vastly exceed our estimate 3×10^{-17} s for $1/2\omega_p$.

and modified Lorentzians

$$f(t)=\frac{n\sin(\pi/2n)}{\pi T}\frac{1}{(1+(t/T)^{2n})} \tag{4.9}$$

which approximate the square step when n is large.

The limit $T\to 0$ clearly reduces all these fs to delta functions, and yields the results that one would have got without time averaging in the first place.

Because of the singularities of the square step (4.6a) at $t=\pm T$, its Fourier transform (4.6b) drops too slowly at infinity to allow $\langle\Delta\bar{S}^2\rangle$ as given by (4.5d) to dispense with the cut-off, and one finds

$$\langle\Delta\bar{S}^2\rangle=\frac{1}{8\pi^4}\frac{\Omega^6}{T^2}\left(1-\frac{\cos 6\theta}{(1+4\Omega^2T^2)^3}\right)\qquad \tan\theta=2T\Omega\qquad \text{(square step).}$$

But for all our other candidates $\langle\Delta\bar{S}^2\rangle$ is well defined in the no-cut-off limit, and takes the form

$$\langle\Delta\bar{S}^2\rangle=\frac{\text{constant}}{T^8} \tag{4.10a}$$

as could have been foreseen on dimensional grounds. The 'constants' here and below are all pure numbers, but different numbers in different equations.

In equation (4.10a) there is no getting away from the fact that the coefficient depends on the shape of f, and no universal numerics can be expected. From here on we adopt the Lorentzian (4.8) in spite of the manifest artificiality which it shares with the other candidates†, because it makes the calculations very easy, partly through allowing $g(\omega+\omega')=e^{-\omega T}e^{-\omega' T}$ to factorize. Accordingly, we evaluate (4.5d) with g given by (4.8b) but without the cut-off (see footnote on p. 7), i.e. replacing $\exp(-\xi/\Omega)\to 1$. This yields

$$\langle\Delta\bar{S}^2\rangle=\frac{21}{2^9\pi^4T^8}\qquad \text{(Lorentzian).} \tag{4.10b}$$

It is now straightforward to take the final step of averaging over $\boldsymbol{r}_\parallel$ as well as over t, as prescribed by (1.3). To this end one needs the two-dimensional Fourier transform $\gamma(\boldsymbol{k})$ of $\phi(\boldsymbol{r}_\parallel)$:

$$\gamma(\boldsymbol{k})\equiv\iint_{-\infty}^{\infty}\mathrm{d}^2r_\parallel\,\phi(\boldsymbol{r}_\parallel)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\parallel). \tag{4.11}$$

Analogously to (4.5) it follows at once that the physical quantity central to our discussion, namely $\langle\Delta\bar{\bar{S}}^2\rangle$ as defined by (1.4), is given by

$$\langle\Delta\bar{\bar{S}}^2\rangle=\sum_{\lambda\lambda'}|\langle\lambda\lambda'|S_{33}|0\rangle|^2|g(\omega+\omega')|^2|\gamma(\boldsymbol{k}+\boldsymbol{k}')|^2. \tag{4.12}$$

† An obvious next step towards greater realism might be to explore the modified Lorentzians (4.9) with large n.

The implications are clearest from the form that emerges by comparison with (4.3c), after introducing dimensionless vector variables $\boldsymbol{K}T \equiv \boldsymbol{\kappa}$, $K \equiv \kappa/T$, etc. Eventually one finds

$$\langle \Delta \tilde{S}^2 \rangle = \frac{1}{T^8} \frac{1}{16\pi^6} \int \mathrm{d}^3\kappa\, \kappa\, \mathrm{e}^{-2k} \int \mathrm{d}^3\kappa'\, \kappa'\, \mathrm{e}^{-2\kappa'}$$
$$\times \{\cos^2\theta \cos^2\theta' \cos^2(\phi-\phi') + \cos^2\theta \sin^2(\phi-\phi')$$
$$+ [-\sin\theta \sin\theta' + \cos(\phi-\phi')]^2\}$$
$$\times \left| \int \mathrm{d}^2 r_\parallel\, \phi(\boldsymbol{r}_\parallel) \exp[-\mathrm{i}(\boldsymbol{\kappa}+\boldsymbol{\kappa}')_\parallel \cdot \boldsymbol{r}_\parallel / cT] \right|^2 \tag{4.13}$$

where (θ, ϕ) and (θ', ϕ') are the polar angles of $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ respectively.

The final factor $|\gamma|^2$ of the integrand in (4.13) depends on the shape of the surface-averaging function ϕ, and without restricting ϕ no significant simplifications can be expected. But (4.13) does simplify crucially in most practically interesting cases because the region where ϕ is appreciable will then be characterized by a maximum typical linear dimension a far shorter than the distance cT that light travels in the observation period T. Since the exponential $\exp[-2(\kappa+\kappa')]$ effectively limits κ and κ' to order unity, the exponent $[-\mathrm{i}(\boldsymbol{\kappa}+\boldsymbol{\kappa}')_\parallel \cdot \boldsymbol{r}_\parallel/cT]$ can then be well approximated by zero. If so, then the final factor in (4.13) reduces to unity by virtue of the norming condition (1.3b) on ϕ; in other words, *once we have averaged over times of order T, further surface averaging over regions with linear dimensions $a \ll cT$ makes no difference.* The conclusion is by no means peculiar to specifically Lorentzian time averaging: the basic physical reason is simply that the time average is dominated by normal modes with frequencies below $1/T$, whose wavelengths in turn are greater than cT, and whose effects therefore are fully coherent over sampling distances much less than this.

Accordingly, the most important result of the present section, to be viewed jointly with (4.10), reads

$$\langle \Delta \tilde{S}^2 \rangle \approx \langle \Delta \bar{S}^2 \rangle = \frac{\text{constant}}{T^8} \qquad \left(\frac{a}{cT} \ll 1 \right). \tag{4.14}$$

Corrections are of relative order $(a/T)^2$; but we repeat that the value of the constant depends on the shape of the time-averaging function.

The expressions (4.12), (4.13) simplify also in the opposite extreme $a \gg cT$, provided ϕ has a reasonably flat central region. (Here a is the *minimum* typical linear dimension characterizing ϕ.) We settle for the sketch of a rough plausibility argument couched explicitly in terms of Lorentzian time averaging. It proves convenient to revert to the integration variables $(l, \boldsymbol{k})$, etc., i.e. to start from

$$\langle \Delta \tilde{S}^2 \rangle = \frac{1}{16\pi^6} \int \mathrm{d}l\, \mathrm{d}^2k\, \omega\, \mathrm{e}^{-2\omega T} \int \mathrm{d}l'\, \mathrm{d}^2k'\, \omega'\, \mathrm{e}^{-2\omega' T} \{\ldots\}$$
$$\times \left| \int \mathrm{d}^2 r_\parallel\, \phi(\boldsymbol{r}_\parallel) \exp[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}') \cdot \boldsymbol{r}_\parallel] \right|^2 \tag{4.15}$$

where the contents of the braces are the same as in (4.13). Since $T \ll a$ by assumption, the exponentials $\mathrm{e}^{-2\omega T}$ and $\mathrm{e}^{-2\omega' T}$ decrease slowly on the scale of the oscillations of $\exp[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}') \cdot \boldsymbol{r}_\parallel]$, whence the last factor in the integrand is approximated first. With

ϕ slowly varying on the relevant scale, we have†

$$\left|\int \mathrm{d}^2 r_{\|}\, \phi \exp[\ldots]\right|^2$$

$$\approx \phi(0) \int \mathrm{d}^2 r_{\|} \exp[\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot \boldsymbol{r}_{\|}] \int \mathrm{d}^2 \boldsymbol{r}_{\|}\, \phi(\boldsymbol{r}) \exp[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot \boldsymbol{r}_{\|}] \tag{4.16a}$$

$$\approx \frac{1}{a^2}(2\pi)^2 \delta^{(2)}(\boldsymbol{k}+\boldsymbol{k}') \int \mathrm{d}^2 r_{\|}\, \phi \exp[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot \boldsymbol{r}_{\|}] \tag{4.16b}$$

$$= \frac{1}{a^2}(2\pi)^2 \delta^{(2)}(\boldsymbol{k}+\boldsymbol{k}') \int \mathrm{d}^2 r_{\|}\, \phi \tag{4.16c}$$

$$= \frac{1}{a^2}(2\pi)^2 \delta^{(2)}(\boldsymbol{k}+\boldsymbol{k}'). \tag{4.16d}$$

In going from (4.16a) to (4.16b) we have estimated $\phi(0) \approx 1/a^2$ on dimensional grounds; the exponent in (4.16b) vanishes by virtue of the prefacing delta function; and the resulting integral in (4.16c) then reduces to unity by virtue of the normalization of ϕ. On substituting (4.16d) into (4.15), we obtain an explicit factor $1/a^2$, while the other factor depends only on T; hence dimensional constraints alone yield the end result

$$\langle \Delta \bar{\tilde{S}}^2 \rangle = \frac{\text{constant}}{a^2 T^6} \qquad \left(\frac{a}{cT} \gg 1\right) \tag{4.17}$$

to be compared with (4.14). By the light of the remark just below (4.9), it is clear that surface averaging without time averaging, which corresponds to finite a but $T \to 0$, is insufficient to ensure convergent, i.e. cut-off-independent, fluctuations.

Note finally that the mean-square time- and surface-averaged force per unit area on our test piston is given by *twice* $\langle \Delta \bar{\tilde{S}}^2 \rangle$, because the two sides of the piston contribute equally.

5. Fluctuations with parallel mirrors

Given two infinitely extended parallel mirrors with separation L, a test piston with typical linear dimensions a, and an averaging time T, one is in principle faced by several different regimes depending on the relative magnitudes of cT, L and a. But for simplicity we shall consider only the realistic case $cT \gg a$; then, as for a single mirror, surface averaging is redundant, and we need determine only the time-averaged fluctuation $\langle \Delta \bar{S}^2 \rangle$.

If $L \gg cT$, we must of course recover the results of section 4. However, this mathematical limiting case is somewhat deceptive, as shown by the scenario widely favoured by theorists doing thought experiments on the Casimir effect. This scenario features *three* parallel mirrors, and one determines the force on the central mirror, separated from the others by intermediate regions of widths L_1 and L_2 respectively.

† The argument is familiar from time-dependent perturbation theory, where one commonly sets $[\delta(\text{frequency})]^2 = \delta(\text{frequency}) \times (\text{duration})/2\pi$. If $\phi = 1/a^2$ over an arbitrarily large conventionally-shaped region of area a^2 (e.g. an ellipse with excentricity fixed independently of a), then (4.15) can be evaluated in closed form. Enough obstinacy with Bessel functions eventually yields (4.17) with constant $= 35/12\pi^3 \approx 0.0941$.

When $L_2/L_1 \gg 1$, the mean force effectively depends on L_1 alone, and can be calculated as if the intermediate region having the larger width L_2 were a half-space, i.e. in the limit $L_2 \to \infty$. But as regards our fluctuations, which depend crucially on T, the mathematical limit $L_2 \to \infty$ is physically appropriate only if $L_2 \gg cT$, which would rarely be true. Accordingly, we shall concentrate mainly on the opposite regime, where both L_1 and L_2 are much smaller than cT. Luckily, here too it turns out that, if $cT \gg L_2 \gg L_1$, the fluctuations of the time-averaged force on a two-sided piston set into the centre mirror are dominated by the fields in the narrower intermediate region, and depend only on L_1.

With these preliminaries understood, we revert until further notice to the forces on just one side of our test piston, situated at $z=0$, and facing inwards into the region $0 \leqslant z \leqslant L$ between just one pair of mirrors.

The expansions (3.2) and (3.3) are now modified by replacing the continuous wavenumber l by $n\pi/L$, $n=0,1,2,\ldots$; and integrals $(1/\sqrt{\pi})\int_0^\infty \mathrm{d}l\ldots$ by $(1/\sqrt{L})\sum_{n=0}^{\infty\,\prime\prime}\ldots$, where the double prime prescribes a tacit factor $2^{-1/2}$ when $n=0$. Correspondingly, in the commutators (3.2c) one now replaces $\delta(l-l')$ by $\delta_{nn'}$; and in sums over *squared* normal-mode amplitudes (the only kind to feature below) $(1/\pi)\int_0^\infty \mathrm{d}l\ldots$ is replaced by $(1/L)\sum_{n=0}^{\infty\,\prime}$, where the single prime assigns to the term with $n=0$ a factor $\frac{1}{2}$, not written explicitly but important to (5.3) and (5.4). These correspondences between slab and half-space are discussed and illustrated extensively elsewhere (Barton and Fawcett 1988, section 10).

For the vacuum expectation value of the stress, (4.2a) thus adapted yields

$$\langle S\rangle = \frac{1}{8\pi^2 L}\sum_n{}' \int \mathrm{d}^2k\,\frac{1}{\omega}\left[k^2 - \left(\frac{n^2\pi^2}{L^2}+\omega^2\right)\right]\mathrm{e}^{-\omega/\Omega}$$

$$= -\frac{\pi}{2L^3}\sum_n n^2 \int_{n\pi/L}^{\infty} \mathrm{d}\omega\;\mathrm{e}^{-\omega/\Omega}$$

$$\langle S\rangle = -\frac{\pi\Omega}{2L^3}\sum_n n^2\,\mathrm{e}^{-\alpha n} \qquad \alpha \equiv \pi/\Omega L. \tag{5.1a, b}$$

In the no-cut-off limit $\alpha \to 0$ this becomes

$$\langle S\rangle = -\frac{\Omega^4}{\pi^2} + \frac{\pi^2}{240L^4} + \mathcal{O}\left(\frac{1}{\Omega^2 L^6}\right). \tag{5.2}$$

The first term reproduces as it must the single-mirror result (4.2b), and the second term is the mean Casimir stress.

The mean-square deviation of the finite-time-averaged stress is obtained by similarly adapting (4.5). We continue with Lorentzian averages (4.8), drop the cut-off, and introduce a convenient scaled variable $\nu \equiv \omega L/\pi$.

A few straightforward steps lead to

$$\langle \Delta\bar{S}^2\rangle = \frac{\pi^4}{4L^8}\sum_n{}'\sum_{n'}{}' \int_n^\infty \mathrm{d}\nu \int_n^\infty \mathrm{d}\nu'\,\exp\left[-\left(\frac{2\pi T}{L}\right)(\nu+\nu')\right]$$
$$\times\left(\tfrac{3}{2}\nu^2\nu'^2 - \tfrac{3}{4}\nu^2 n'^2 - \tfrac{3}{4}\nu'^2 n^2 + \tfrac{3}{2}n^2 n'^2\right). \tag{5.3}$$

In the mathematical limit $L \to \infty$, i.e. $T/L \to 0$, the sums over n and n' reduce to integrals, and one eventually recovers the single-mirror result (4.10b). But for the reasons explained at the start of this section we confine attention from here on to the opposite regime $T/L \gg 1$. Then (5.3) is dominated by the terms with $n=0=n'$, up to corrections of relative order $\exp(-T/L)$. Keeping only $n=0=n'$, and remembering the factors $\frac{1}{2}$ prescribed by the prime on each sum, we find

$$\langle \Delta \bar{S}^2 \rangle = \frac{\pi^4}{4L^8}\frac{1}{4}\frac{3}{2}\left[\int_0^\infty d\nu\, \nu^2 \exp\left(-\frac{2\pi T\nu}{L}\right)\right]^2$$

$$= \frac{3}{2^9\pi^2L^2T^6} \qquad \text{(Lorentzian, } cT \gg L\text{)} \tag{5.4a}$$

while in the general case we can assert only

$$\langle \Delta \bar{S}^2 \rangle = \frac{\text{constant}}{L^2T^6} \qquad (cT \gg L). \tag{5.4b}$$

These results should be compared with (4.10), and, with equal legitimacy under the condition $a/T \ll 1$, with (4.14). We see that fluctuations between parallel mirrors *are enhanced relative to those in a half-space* by the large factor $(T/L)^2$. This may be thought surprising, because folklore has it that geometric restrictions produce frequency gaps (as indeed they do here in branches with non-zero n), and that such gaps tend to suppress excitations. The present writer knows of no resolution of this paradox (if paradox it is) in words of one syllable.

More interesting perhaps than the absolute magnitudes are the root-mean-square *fluctuations relative to the mean Casimir stress*, namely the ratio

$$\frac{\langle \Delta \bar{S}^2 \rangle^{1/2}}{(\pi^2/240L^4)} \approx \frac{(1/L^2T^6)^{1/2}}{(1/L^4)} = \left(\frac{L}{cT}\right)^3 \ll 1. \tag{5.5}$$

Suppose that $T=10^{-8}$ s, which though optimistic for a force detector is perhaps not totally absurd, and that $L=0.1$ mm. Then $(L/cT)^3 \approx 4\times10^{-14}$, outrageously far below the accuracy of any practicable apparatus.

The situation for the three-mirror scenario can now be summarized quite succinctly in the regime where $T \gg a$ and $T \gg L_2 \gg L_1$. We are concerned with the random variable $\bar{S}$, redefined here as the total force per unit area on a test piston set in the centre mirror, averaged over times of order T. The mean value is $\langle \bar{S} \rangle = \langle S \rangle = \pi^2/240L^4$, directed towards the nearer of the two flanking mirrors. This result needs no time averaging, and does not depend on how any such averaging is implemented; but it does represent the finite difference between two oppositely directed forces (one exerted on each side of the piston), which taken separately diverge for perfect mirrors, and for imperfect mirrors are of order $\Omega^4 \approx \omega_p^4$ as shown by (5.2). The mean-square deviation is $\langle \Delta \bar{S}^2 \rangle =$ constant$/L_1^2T^6$. The value of the constant does depend on just how the time averaging is implemented; it is convergent even without a cut-off, and is very insensitive to mirror imperfections provided $T\omega_p \gg 1$; and it stems from zero-point fluctuations of the field *intensities* in the narrower of the two intermediate regions.

Accordingly, the measurable fluctuations of the Casimir stress even for the more realistic two-mirror scenario can be calculated quite safely by considering the interior region alone. This conclusion is not wholly trivial, because if one did need to estimate the (far weaker) fluctuations of the stress exerted on the outer piston surface, then the half-space formula (4.14) would probably probe quite unreliable: the point is that laboratory apparatus is likely to have other conductors separated from the mirrors by distances less than cT.

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Appendix. On zero-point probability distributions

By contrast with the body of this paper, where we calculated only mean-square deviations, we now consider some rudiments of the underlying probability distributions, which, once determined, would naturally carry much more information. For simplicity (and without any loss of essentials) we consider only unbounded space, and start with the appropriate normal-mode expansions of $\boldsymbol{E}$ and $\boldsymbol{B}$, obtainable as explained *à propos* of (3.1) above. Dimensional reasons alone suffice to make the cut-off prominent, in the first of its two roles explained at the end of section 3.

The vacuum expectation values of the fields vanish, but those of any one squared Cartesian field component, say of $E_j(\boldsymbol{r}, t)$, read

$$\langle E_j^2\rangle = \sum_\lambda |E_{\lambda_j}|^2 = \sum_s \int \mathrm{d}^3K \frac{K}{4\pi^2}\frac{1}{3}\mathrm{e}^{-K/\Omega}$$

$$= \frac{2}{3\pi}\int_0^\infty \mathrm{d}K\, K^3\, \mathrm{e}^{-K/\Omega} = \frac{4\Omega^4}{\pi}. \qquad \text{(A1)}$$

Strictly speaking, the mere fact that this diverges in the no-cut-off limit does not automatically taboo the underlying probability distribution $\mathbb{P}_{E_j}(v)$, defined so that $\mathbb{P}_{E_j}(v)\,\mathrm{d}v$ is the probability of obtaining a result within $\mathrm{d}v$ of v when one measures E_j *in vacuo*. For instance, a Lorentzian $\mathbb{P}$ could be perfectly well defined and yet imply a divergent $\langle E_j^2\rangle = \int_{-\infty}^{\infty} \mathrm{d}v\, \mathbb{P}_{E_j}(v)v^2$.

Luckily, in this simplest of cases $\mathbb{P}_{E_j}$ is readily determined via its characteristic function (Fourier transform) $F_{E_j}(u)$, defined by

$$F_{E_j}(u) \equiv \langle \exp(\mathrm{i}uE_j)\rangle = \int_{-\infty}^{\infty} \mathrm{d}v\, \mathbb{P}_{E_j}(v)\exp(\mathrm{i}uv) \qquad \text{(A2}a\text{)}$$

$$\mathbb{P}_{E_j}(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}u\, F_{E_j}(u)\exp(-\mathrm{i}uv). \qquad \text{(A2}b\text{)}$$

To calculate F, we first define the positive- and negative-frequency parts $E_j^{(\pm)}$ or E_j by

$$E_j^{(+)} = \sum_\lambda a_\lambda E_{\lambda_j} \qquad E_j^{(-)} = \sum_\lambda a_\lambda^+ E_{\lambda_j}^* \qquad \text{(A3)}$$

and recall the normal-ordering Baker–Hausdorff formula

$$\exp(\mathrm{i}uE_j) = \exp(\mathrm{i}uE_j^{(-)})\exp(\mathrm{i}uE_j^{(+)})\exp\{-\tfrac{1}{2}[\mathrm{i}uE_j^{(-)}, \mathrm{i}uE_j^{(+)}]\} \qquad \text{(A4)}$$

applicable because $E_j^{(\pm)}$ commute with their commutator

$$[E_j^{(-)}, E_j^{(+)}] = -\sum_\lambda |E_{\lambda_j}|^2 = -\langle E_j^2\rangle. \qquad \text{(A5)}$$

Since the vacuum expectation-value of the operator on the right of (A4) is unity, we have

$$F_{E_j}(u)=\exp(-\tfrac{1}{2}u^2\langle E_j^2\rangle)=\exp\left(-\frac{2u^2\Omega^4}{\pi}\right) \tag{A6a}$$

$$\mathbb{P}_{E_j}(v)=(2\pi\langle E_j^2\rangle)^{-1/2}\exp\left(-\frac{v^2}{2\langle E_j^2\rangle}\right). \tag{A6b}$$

Now, simply because the distribution is Gaussian (as a direct consequence of the Baker–Hausdorff formula), it is fully specified by its mean-square deviation $\langle E_j^2\rangle$ alone; therefore it does become wholly unacceptable as Ω and $\langle E_j^2\rangle$ diverge. We must conclude that, in any ordinary statistical acceptation, E_j itself is not a physical observable at all.

On the other hand, if we average E_j over time, say with the same Lorentzian as in (4.8), then we do get a perfectly acceptable Gaussian for $\bar{E}_j$, with width proportional to T^{-2}.

The distributions of the E_j trivially determine those of the E_j^2, which in turn, though less obviously, determine the distribution of $\boldsymbol{E}^2$. One necessary condition for this latter connection to be straightforward is that the variables E_1, E_2, E_3 commute. In fact it turns out that their joint distribution is just the product of their separate distributions, so that they are uncorrelated in the usual statistical sense†.

It is far more of a challenge to determine the distributions of the energy density $\mathscr{H}=(\boldsymbol{E}^2+\boldsymbol{B}^2)/8\pi$, or of the stress S_{33} as given by (1.1), because the E_j fail to commute with the B_i. Such calculations too are probably best tackled via the characteristic functions, with the techniques developed for what today would be called multimode squeezed states, clearly set out say by Agrawal and Mehta (1977). These techniques furnish formulae for the vacuum expectation values of exponentials whose exponents are of at most second order in the annihilation and creation operators a_λ and $a^+_{\lambda'}$. For the Maxwell field, $\mathbb{P}_{\mathscr{H}}$ *remains to be found, though the present writer (unpublished)* has evaluated its analogue for a one-dimensional scalar field: the calculation is manageable because the matrix of the quadratic in the exponent is a sum of (just a few) separable matrices.

Unfortunately, the most immediate *desideratum* suggested by sections 4 and 5 is the probability distribution not of S but of its time average, $\bar{S}$, which presumably remains well defined even in the no-cut-off limit. However, even for the one-dimensional scalar model there seems to be no known way of evaluating the standard formula for the characteristic function of $\bar{S}$, because the quadratic form in the exponent of $\exp(iu\bar{S})$ is not a sum of a finite number (let alone of just a few) separable parts. Thus the question of the probability distribution of $\bar{S}$ remains open.

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† One should not jump to conclusions. The mere fact that two variables commute does not ensure that they are uncorrelated in (say) the vacuum state. Witness for instance $E_j(\boldsymbol{r},t)$ and $E_i(\boldsymbol{r}',t')$ at space-like separation. They commute, yet they are correlated, because $\langle E_j(\boldsymbol{r},t)E_i(\boldsymbol{r}',t')\rangle\neq\langle E_j(\boldsymbol{r},t)\rangle\langle E_i(\boldsymbol{r}',t')\rangle=0$.

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$\backslash \quad 2\langle E_j^2\rangle)/$

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